# Tailoring symmetry groups using external alternate fields.

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Macroscopic systems with continuous symmetries subjected to oscillatory fields have phases and transitions that are qualitatively different from their equilibrium ones. Depending on the amplitude and frequency of the fields applied, Heisenberg ferromagnets can become XY or Ising-like — or, conversely, anisotropies can be compensated — thus changing the nature of the ordered phase and the topology of defects. The phenomenon can be viewed as a dynamic form of *order by disorder*.

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The effect of alternate fields on interacting many-body systems has been a subject of long-standing interest. In particular, magnetic hysteresis has been extensively studied for clean or disordered Ising-like systems [1–5].

A much less studied problem is the effect of a periodic drive on systems having a continuous symmetry either in a pure or a disordered [7–9,6] case; even though, as we shall see, the existence of soft modes make the systems respond in striking and interesting ways. Years ago, Rao *et al.* [7,8] and Dhar and Thomas [9] studied a driven clean $N$-component Heisenberg ferromagnet, solvable analytically in the large-$N$ limit for any dimension $d$. Due to the fact that at $N = \infty$ almost any configuration is perpendicular to the field, one can not study thus, phenomena that depend on a delicate balance between a tendency of the system to order longitudinally or transversely to the field.

In this paper we explore the full problem for $d > 2$ and $N \geq 2$. We firstly treat the XY ($N = 2$) model and show that, surprisingly enough, there are in fact *three* kinds of ferromagnetic phases, with the magnetization longitudinal, transverse and canted with respect to the field, respectively. We show this directly for the mean-field ($d = \infty$) case, and extend the result for all dimensions $d > 2$ by use of low frequency and low temperature expansions. The same analytic methods show that for the Heisenberg case ($N \geq 3$) there is always transverse order [9], again in all $d > 2$. In addition we show that one can tailor in this case the symmetry group from $O(3)$ to $O(2)$ or to $Z_2$ by applying one or more a.c. fields: a possibility we argue is quite general for systems with continuous symmetries.

Our motivation is that an understanding of this system is a base to explore the connections with wider class of forced systems with continuous symmetries, including liquid crystals [10], lamellar polymers [11], crystalline defects, ferromagnetic conductors [12], etc.

Consider the $O(n)$ ferromagnet

$$E_{int} = -\frac{1}{2d}\sum_{ij} \vec{S}_i \cdot \vec{S}_j \qquad (1)$$

where $2d$ is the number of neighbors, itself depending on lattice dimensionality and topology. The $\vec{S}$ are $N$-dimensional vectors either of fixed norm *(hard spins)*, or whose length may fluctuate according to a soft-spin term. The system is coupled to an ac field:

$$E_h = -\cos(\omega t) \sum_i \vec{h} \cdot \vec{S}_i \qquad (2)$$

and we shall consider the dynamics in the *strongly dissipative limit*:

$$\dot{S}_i^\alpha = -\frac{\partial E}{\partial S_i^\alpha} + \eta_i^\alpha \qquad (3)$$

where $\eta_i^\alpha$ is a white gaussian noise of variance $2T$. The energy $E$ is the sum of (1), (2) and a *soft-spin* term fixing the spin length or, alternatively, a Lagrange multiplier for *hard* spins. In (3), we are neglecting precessional effects (which bring in a host of interesting phenomena [13]).

**Selection: order by disorder.**

Consider the 'hard' spin model at zero temperature, subjected to an a.c. field. We shall use the example of the XY model, but the argument is valid for any number of components and in any number of dimensions. Using polar coordinates, we have for the angle $\theta_i$:

$$\dot{\theta}_i = -\frac{1}{d}\sum_j \sin(\theta_i - \theta_j) - h\cos(\omega t)\cos\theta_i \qquad (4)$$

This always admits a solution in which all spins move in phase $\theta_i = \theta, \quad \forall i$ (inhomogeneous solutions decay into this one) which makes interaction terms zero and we have for any geometry:

$$\theta(t) = 2\tan^{-1}\left(e^{-[\frac{h}{\omega}\sin(\omega t)+k]}\right) - \frac{\pi}{2} \qquad (5)$$

By considering all possible values of the integration constant $k$, we conclude that at zero temperature solutions are possible in which the total magnetization vector oscillates around *any possible angle*, and without hysteresis. Unlike the case without field, these solutions are no

longer related by a continuous symmetry, and only the discrete symmetries $\theta \to -\theta$ and $\theta \to \pi - \theta$ remain. (In the case of $N > 2$, a symmetry subgroup survives: for example, in $N = 3$ the spins evolve with $\theta$ as in (5) and with constant $\varphi$; solutions with the same $k$ but different $\varphi$ are related by an $O(2)$ symmetry.)

In situations like this one, when one has at $T = 0$ solutions that are unrelated by a symmetry, thermal or quantal fluctuations will generically select a subset of them (unless of course they destroy the order altogether). This phenomenon has been named *order by disorder* [14], as it is the fluctuations that are responsible for a reduction in the number of solutions. In practice, if we prepare the system at small $T$ in one of the possible $T = 0$ solutions, it will drift to the selected average angle: there is a secular perturbation acting on timescales much slower than the vibration frequency. Temperature also brings in hysteresis and hence dissipation.

**Paramagnetic, transverse, longitudinal and canted solutions.**

In an $O(N)$ ferromagnet, one can have four types of phases, which can be best distinguished by considering the magnetization in the direction parallel and perpendicular to the field $M_h(t)$ and $M_\perp(t)$, the angle $\theta(t) = \tan^{-1}(M_\perp(t)/M_h(t))$, and the corresponding averaged quantities: $\overline{M^\alpha} \equiv \frac{1}{\tau}\oint dt M^\alpha(t)$ and $\bar\theta \equiv \frac{1}{\tau}\oint dt\, \theta(t)$

- *Paramagnetic*: The magnetization follows the field with a delay (hysteresis): $M_\perp(t) = 0$ and $\overline{M_h} = 0$.
- *Longitudinal* ($\theta(t) = 0$ *or* $\theta(t) = \pi$): the magnetization points in the direction of the field: $M_\perp(t) = 0$, but $\overline{M_h} \neq 0$.
- *Transverse* ($\bar\theta = \pi/2$ *or* $\bar\theta = -\pi/2$)*:* the magnetization has a non-zero component $M_\perp(t) \neq 0$ orthogonal to the field, the component parallel to the field has zero time-average $\overline{M_h} = 0$.
- *Canted* ($0 < \bar\theta < \pi/2$ *or* $\pi/2 < \bar\theta < \pi$)*:* The magnetization evolves around an oblique angle with the field's direction: $\overline{M_\perp} \neq 0$ and $\overline{M_h} \neq 0$.

The longitudinal solution is doubly degenerate. In the XY case, the canted solutions have degeneracy four and the transverse ones two. Both of them are continuously degenerate for $N > 2$ (one solution per plane determined by $(M_h, M_\perp)$).

Dynamical phase transitions to a magnetized phase are well attested in the case of an Ising system [15–17]. For continuous $O(N = \infty)$ systems, Rao *et al* ( [7,8]) found a dynamical phase transition on increasing frequency from a paramagnetic state with $\overline{M} = 0$ to a ferromagnetic regime. Subsequently, Dhar and Thomas [9] pointed out that the order is in fact always transverse, and $\overline{M_h} = 0$ [18]. They also studied the case $N \geq 2$ in finite dimensions within the unmagnetized phase [19,20].

**Dynamic mean-field approximation.**

The dynamic mean-field approximation has the advantage of being completely solvable, and one can easily get a complete phase diagram. The equations consist of a single-spin equation and a self-consistency condition:

$$\begin{aligned} \dot S^\alpha &= (M^\alpha + h^\alpha)(t) - \lambda(t) S^\alpha + \eta^\alpha \\ M^\alpha(t) &= \langle S^\alpha(t)\rangle_{single\ spin} \end{aligned} \tag{6}$$

where $\lambda$ is a Lagrange multiplier imposing the spin length. One can solve the system (6) by considering the evolution of the set of expectation values. For example, for the XY model, using the Fokker-Plank equation associated with (6) we write an exact infinite system of equations for $x_n \equiv \langle\cos(n\theta)\rangle$, $y_n \equiv \langle\sin(n\theta)\rangle$, $n = 1, ...$:

$$\begin{aligned} \dot y_n &= -n^2 T y_n + \frac{1}{2} n x_1 (y_{n-1} - y_{n+1}) + m(x_{n-1} + x_{n+1}) \\ \dot x_n &= -n^2 T x_n + \frac{1}{2} n x_1 (x_{n-1} - x_{n+1}) - m(y_{n-1} + y_{n+1}) \\ \dot y_1 &= (\frac{1}{2} - T) y_1 - \frac{1}{2} x_1 y_2 - \frac{1}{2}(y_1 + h(t)) x_2 + \frac{1}{2} h(t) \\ \dot x_1 &= (\frac{1}{2} - T) x_1 - \frac{1}{2} x_1 x_2 - \frac{1}{2}(y_1 + h(t)) y_2 \end{aligned} \tag{7}$$

with $m(t) = n(y_1 + h(t))/2$. For the Heisenberg case one has to study the dynamics of the expectation values of the spherical harmonics $Y_{lm}$.

We have numerically solved the system (7), keeping as many modes as necessary, for various values of $h, \omega, T$. The main results are summarized in the figures.

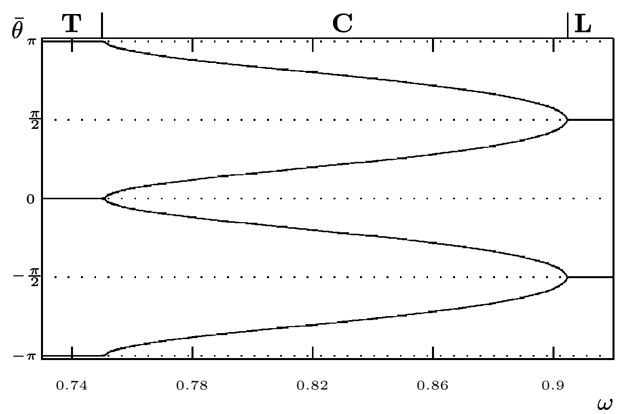


FIG. 1. $\bar\theta$ as a function of $\omega$ for the XY mean field model, $h = 1$, $T = 0.2$. The transitions are second order.

FiG. 1. shows the value of $\bar\theta$ vs $\omega$ at temperature $T = 0.2$ and $h = 1$. We see second order transitions, as frequency decreases, from longitudinal (L) to canted (C), and from canted to transverse (T). FIG. 2. gives the general phase diagram of the system at $h = 1$ (the dashed line at T=0.5 represents the para-ferro transition at zero field). In the limit of high frequencies $\omega \to \infty$, $h/\omega \to 0$, there is a dynamical critical temperature $T_d \sim .42$ from a longitudinal to a transverse phase.

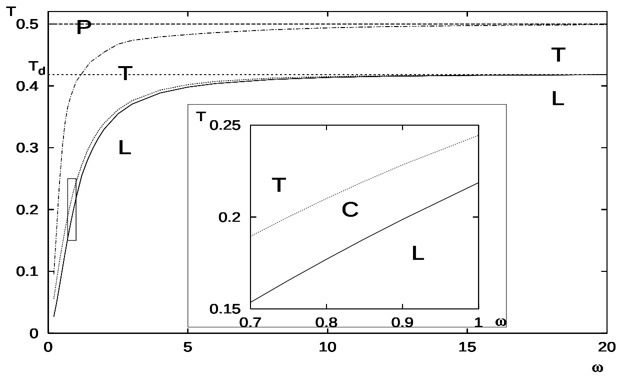


FIG. 2. $T$-$\omega$ Phase diagram for the XY mean field model, $h = 1$.

In the Heisenberg case, we have only found paramagnetic or transverse behavior. Below we confirm this by showing that, in all dimensions $d \geq 3$, no longitudinal order appears in the first order in low-temperature expansion (the most favourable case, as seen in Fig. 2), thus supporting the conjecture of [9] that the Heisenberg model, and *a foriori* all the $N > 2$ cases, are in this sense qualitatively different from the XY model. However, as we shall see below, *one can still select a single direction* in a Heisenberg case, but with the application of *two* fields.

Let us stress here that at very low frequencies mean-field is potentially very misleading, as the physics may be dominated by nucleation effects which this approximation neglects.

**Weak fields and low frequencies: a general mechanism for transverse selection.**

In the limit of low frequencies and weak fields ($\omega \to 0$, $h/\omega$ constant), a general simple argument can be given to show that there is transverse selection for any $N \geq 3$ and $d \geq 3$: Consider a field applied along the $z$ axis, and split it in components aligned $h_M(t)$ and normal $h_\perp(t)$ to the instantaneous magnetization. Because by assumption the field is weak and varies slowly, the effect of the aligned component of the field is to change the magnetization norm $M$ linearly and adiabatically: $M(t) = M_o + C h_M(t)$, where $M_o$ is the unperturbed norm and $C$ a positive susceptibility. On the other hand, the effect of the perpendicular field is to modify the angle $\Omega$ between the magnetization vector and the equatorial plane as: $J(M)\dot{\Omega} = h_\perp(t)$. The mobility coefficient $J^{-1}(M)$ depends only upon the norm $M$ due to symmetry. Putting this together we get:

$$J(M)\dot{\Omega} = h(t)\cos\Omega \quad ; \quad M = M_o + C h(t) \sin\Omega \tag{8}$$

and using the weak field approximation:

$$\dot{\Omega} = \frac{h(t)\cos\Omega}{J(M_o) + C J'(M_o) h(t) \sin\Omega} \tag{9}$$

It is easy to check that this equation yields transverse selection provided $J'(M_o) > 0$, i.e. when the mobility decreases with the magnetization. Let us note that an inhomogeneous nucleation mechanism would need a critical droplet size of $h^{-1/2}$, and a free-energy barrier $\sim h^{-(d-2)/2}$, which in the present regime requires much longer times compared to $\omega^{-1}$.

We can now explain in words the selection as a ratchet mechanism: suppose the magnetization starts at an angle in the first quadrant. During the semicycle when the magnetic field points upwards it simultaneously rotates the magnetization towards the $z$ axis and it stretches the norm. In the negative cycle, the rotation is towards the $xy$ plane and the norm is shortened. The slight changes in the norm make the positive cycle less efficient (if $J'(M_o) > 0$) than the negative cycles, and hence the vector has a net drift towards the transverse direction on each cycle. The same mechanism applies for *soft spins* in all dimensions at zero temperature.

We have checked that indeed $J'(M_o)$ is positive in all cases. Near the critical point $J(M) \propto M$, implying that the selection mechanism dissappears at the paramagnetic transition.

**Low $T$ expansion: longitudinal selection**

These models are obviously exactly solvable at $T = 0$, and an expansion in powers of $T$ is feasible *in any dimension.* In the XY case, one decomposes $\theta_i(t) = \langle \theta_i(t) \rangle + \tilde{\theta}_i$ (averages over the thermal noise), and one writes an evolution equation for the expectation values $\langle \theta_i \rangle(t)$, $\langle \tilde{\theta}_i \tilde{\theta}_j \rangle(t)$, $\langle \tilde{\theta}_i \tilde{\theta}_j \tilde{\theta}_k \rangle(t)$, .... To leading order in $T$ only two-point correlations are necessary. In the most general case, we have:

$$\frac{1}{2}\frac{d}{dt}\langle \tilde{\theta}_a \tilde{\theta}_b \rangle = T\delta_{ab} - \sum_j A_{aj} \langle \tilde{\theta}_b (\tilde{\theta}_a - \tilde{\theta}_j) \rangle - h \langle \tilde{\theta}_a \tilde{\theta}_b \rangle \sin\langle \theta_a \rangle$$

$$\frac{d\langle \theta_a \rangle}{dt} = h(t)(1 - \frac{1}{2}\langle \tilde{\theta}_a^2 \rangle) \cos\langle \theta_a \rangle \tag{10}$$

where $A_{ai} \propto \frac{1}{d}$ if $a, i$ are neighbors in $d$-dimensional space and zero otherwise. This system is linear in the correlations. One can assume translational invariance with one-point functions independent of the site and two-point functions depending only on the distance between sites. The system then becomes exactly solvable in Fourier basis (details in [21]). For hard spin systems, it always yields *longitudinal* selection for any $d \geq 3$. Thus, we have shown the existence of a longitudinal and a transverse phase for the XY-model in finite dimension $d \geq 3$. The existence of the intermediate canted phase we have obtained analytically within mean-field (large $d$) has been checked with simulations.

The generalization to the Heisenberg ($N = 3$) case of the low-temperature expansion is immediate, one has to consider two-point correlations of both angles $\theta_i$ and $\phi_i$ [21]. One can show that there is no selection to order $T$

in any dimension $d \geq 3$. Since the lower temperatures are the most favourable for a longitudinal order, this is evidence that there is no longitudinal phase at all.

**High frequencies**

Another instructive method is the high-frequency expansion in which the external field can be treated perturbatively. We start from a system in equilibrium under an infinitesimal field in a direction $\xi$. In terms of the linear $\chi^{\xi}_{\alpha\beta}$ and higher susceptibilities, the variation of the magnetization vector $M_\alpha$ in presence of the a.c. field $h_\alpha(t)$ is given to second order as:

$$\int dt' \chi^{\xi}_{\alpha\beta}(t,t') h_\beta(t') + \int dt' dt'' \chi^{\xi}_{\alpha\beta\gamma}(t,t',t'') h_\beta(t') h_\gamma(t'')$$

The linear term does not contribute to the drift if $\oint h_\alpha(t) dt = 0$. The quadratic term does and, in the large $\omega$ limit, the contribution reads:

$$\frac{\Delta \overline{M^\alpha}}{\tau} = \frac{1}{\omega^2} \sum_{\beta\gamma} v_{\alpha\beta\gamma} \overline{h_\beta h_\gamma} \;\; ; \;\; v_{\alpha\beta\gamma} \equiv \lim_{\substack{t-t' \to \infty \\ t'' \to t'}} \frac{\partial \chi^{\xi}_{\alpha\beta\gamma}(t,t',t'')}{\partial t'} \tag{11}$$

where $\overline{h_\beta h_\gamma} \equiv \oint h_\beta(t) h_\gamma(t) dt \neq 0$. Specialising to the mean-field case, already the first order in $T$ and in $\omega^{-2}$ gives for the XY case a longitudinal selection corresponding to an effective potential $\sim T h^2 \omega^{-2} \sin(2\theta)$.

Now, if the field is a sum of two components acting at right angles with a ratio of frequencies larger than two, interferences appear to the third order and their contribution is additive up to $\omega^{-2}$. We have checked in the hard spin case that two orthogonal fields at two high enough frequencies $\omega$ and $3\omega$, make the Heisenberg system Ising-like: the magnetization is transverse to one field and longitudinal to the other. The same effect can be obtained with a field with constant modulus that rotates on a plane.

**Perspectives**

One can forsee several applications of these ideas in different fields. Changing the symmetry group of a system leads to a change in the topology of the defects: thus an alternate field may perhaps be used as a tool to study in detail the role these play in the transitions [12], in the phase-ordering kinetics, in the anomalous Hall effect (since it involves the interaction of electrons with the topological defects) [12], etc. Similarly, the *chirality scenario* [22] for the spin-glass transition (which assumes that spin glasses are essentially isotropic) could be put to test experimentally by monitoring the effect of an increased and a decreased effective anisotropy. Subjecting ordered liquid crystals to a.c. fields might be a way to directly observe what becomes of the order and the textures, a strategy for which there are interesting precedents [23].

Such richness of amusing and perhaps useful phenomena is in contrast with the extreme poverty of concepts for these out of equilibrium problems, since there are few qualitative ideas to guide us before having an actual solution, as we have for equilibrium thermodynamic systems.

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